TRIPLE-POINT DEFECTIVE REGULAR SURFACES

LUCA CHIANTINI AND THOMAS MARKWIG

Abstract. In this paper we study the linear series $|L-3p|$ of hyperplane sections with a triple point p of a surface S embedded via a very ample line bundle L for a *general* point p. If this linear series does not have the expected dimension we call (S,L) *triple-point defective*. We show that on a triple-point defective *regular* surface through a general point every hyperplane section has either a triple component or the surface is rationally ruled and the hyperplane section contains twice a fibre of the ruling.

The results of this paper have been generalised in the paper Triple point defective surfaces *(arXiv:0911.1222) by the same authors. Large parts of the latter paper coincide with this paper and the reader should rather refer to that paper than to this one.*

1. Introduction

Throughout this note, S will be a smooth projective surface, $K=K_S$ will denote the canonical class and L will be a divisor class on S such that L and $L-K$ are both *very ample*.

The classical *interpolation problem* for the pair (S,L) is devoted to the study of the varieties:

$$V^{gen}_{m_1,\ldots,m_n}=\big\{C\in|L|\;\big|\;p_1,\ldots,p_n\in S\text{ general, }\operatorname{mult}_{p_i}(C)\geq m_i\big\}.$$

In a more precise formulation, we start from the incidence variety:

$$\mathcal{L}_{m_1,\ldots,m_n}=\{(C,(p_1,\ldots,p_n))\in|L|\times S^n\mid\operatorname{mult}_{p_i}(C)\geq m_i\}$$

Date: 30 November, 2006.

1991 *Mathematics Subject Classification.* 14H10, 14J10, 14C20, 32S15.

The second author was supported by the EAGER node of Torino, and by the Institute for Mathematics and its Applications (IMA), University of Minnesota.

together with the canonical projections:

$$\begin{tikzcd} \mathcal{L}_{m_1,\dots,m_n} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & S^n \\ |L| = \mathbb{P}(H^0(L)^*) \end{tikzcd} \tag{1}$$

As for the map α, the fibre over a fixed point $(p_1, \dots, p_n) \in S^n$ is just the linear series $|L - m_1p_1 - \dots - m_np_n|$ of effective divisors in $|L|$ having a point of multiplicity at least m_i at p_i. These fibres being irreducible, we deduce that if α is *dominant* then $\mathcal{L}_{m_1,\dots,m_n}$ has a unique irreducible component, say $\mathcal{L}^{gen}_{m_1,\dots,m_n}$, which dominates S. The closure of its image

$$V_{m_1,\dots,m_n} := V_{m_1,\dots,m_n}(S, L) := \overline{\beta(\mathcal{L}^{gen}_{m_1,\dots,m_n})}$$

under β is an irreducible closed subvariety of $|L|$, a *Severi variety* of (S, L).

Imposing a point of multiplicity m_i corresponds to killing $\binom{m_i+1}{2}$ partial derivatives, so that

$$\dim |L - m_1p_1 - \dots - m_np_n| \geq \max \left\{ -1, \dim |L| - \sum_{i=1}^{n} \binom{m_i + 1}{2} \right\},$$

and we expect that the previous inequality is in fact an equality, for the choice of general points $p_1, \dots, p_n \in S$.

When this is not the case, then the surface is called *defective* and is endowed with some special structure.

The case when $m_i = 2$ for all i has been classically considered (and solved) by Terracini, who classified in [Ter22] double–point defective surfaces. In any event, the first example of such a defective surface which is smooth is the Veronese surface, for which $n = 2$.

It is indeed classical that imposing multiplicity two at a general point to a very ample line bundle $|L|$ always yields three independent conditions, so that $\dim |L - 2p| = \dim |L| - 3$ and the corresponding Severi variety has codimension 1 in $|L|$.

Furthermore, when S is double-point defective, then any general curve $C \in |L - 2p_1 - \dots - 2p_n|$ has a double component passing through each point p_i.

When the multiplicities grow, the situation becomes completely different. Even in the case $S = \mathbb{P}^2$, the situation is not understood and there are several, still unproved conjecture on the structure of defective embeddings (see [Cil01] for an introductory survey).
When S is a more complicated surface, it turns out that even imposing just one point of multiplicity 3, one may expect to obtain a defective behaviour.

Example 1
Let $S = \mathbb{F}_e \xrightarrow{\pi} \mathbb{P}^1$ be a Hirzebruch surface, $e \geq 0$. We denote by F a fibre of π and by C_0 the section of π of minimal self intersection $C_0^2 = -e$ – both of which are smooth rational curves. The general element C_1 in the linear system $|C_0 + eF|$ will be a section of π which does not meet C_0 (see e.g. [FuP00], Theorem 2.5).
Consider now the divisor $L = 2 \cdot F + C_1 = (2+e) \cdot F + C_0$. Then for a general $p \in S$ there are curves $C_p \in |C_1 - p|$ and there is a unique curve $F_p \in |F - p|$, in particular $p \in F_p \cap C_p$. For each choice of C_p we have

$$2F_p + C_p \in |L - 3p|.$$

Since $F.L = 1 = F.(L - F)$ we see that every curve in $|L - 3p|$ must contain F_p as a double component, i.e.

$$|L - 3p| = 2F_p + |C_1 - p|.$$

Moreover, since $p \in S$ is general we have (see [FuP00], Lemma 2.10)

$$\dim |C_1 - p| = \dim |C_1| - 1 = h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\big) + h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e)\big) - 2 = e$$

and, using the notation from above,

$$\dim(V_3) \geq \dim |C_1 - p| + 2 = e + 2.$$

However,

$$\dim |L| = h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)\big) + h^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2+e)\big) - 1 = e + 5,$$

and thus

$$\operatorname{expdim}(V_3) = \dim |L| - 4 = e + 1 < e + 2 = \dim(V).$$

We say, $(\mathbb{F}_e, L)$ is *triple-point defective*, see Definition 2.

Note, moreover, that

$$(L-K)^2 = (4F + 3C_0)^2 = 24 > 16.$$

□

It is interesting to observe that, even though, in the previous example, the general element of $|L - 3p|$ is non reduced, still the map β of Diagram (1) has finite general fibers, since the general element of $|L - 3p|$ has no triple components.

The aim of this note is to investigate the structure of pairs (S, L) for which the linear system $|L-3p|$ for $p \in S$ general has dimension bigger that the expected value $\dim|L| - 6$, or equivalently, the variety $\mathcal{L}_3^{gen}$, defined as in Diagram (1), has dimension bigger than $\dim|L| - 4$.

Definition 2
We say that the pair (S, L) is *triple-point defective* or, in classical notation, that (S, L) *satisfies one Laplace equation* if

$$\dim|L - 3p| > \max\{-1, \dim|L| - 6\} = \operatorname{expdim}|L - 3p|$$

for $p \in S$ general.

Remark 3
Going back to Diagram (1), one sees that (S, L) is triple-point defective if and only if either:

- $\dim|L| \leq 5$ and the projection $\alpha : \mathcal{L}_3 \to S$ dominates, or
- $\dim|L| > 5$ and the general fibre of the map α has dimension at least $\dim|L| - 5$.

In particular, (S, L) is triple-point defective if and only if the map α is *dominant* and

$$\dim(\mathcal{L}_3^{gen}) > \dim|L| - 4.$$

The particular case in which the general fiber of the map β in Diagram (1) is positive-dimensional, (i.e. the general member of V_3 contains a triple component through p) has been investigated in [Cas22], [FrI01], and [BoC05]. We will recall the classification of such surfaces in Theorem 8 below.

Even when β is generically finite, one of the major subjects in algebraic interpolation theory, namely Segre's conjecture on defective linear systems *in the plane*, says in our situation that, when (S, L) is triple-point defective, then the general element of $|L - 3p|$ must be non-reduced, with a double component through p (exactly as in the case of Hirzebruch surfaces).

We are able to show, under some assumptions, that this part of Segre's conjecture holds, even in the more general setting of *regular* surfaces. Indeed our main result is:

Theorem 4

Let S be a regular *surface, and suppose that with the notation in* (1) α *is dominant. Let L be a very ample line bundle on S, such that $L - K$ is also very ample. Assume $(L - K)^2 > 16$ and (S, L) is triple-point defective.*

Then S is rationally ruled in the embedding defined by L. Moreover a general curve $C \in |L - 3p|$ contains the fibre of the ruling through p as fixed component with multiplicity at least two.

Remark 5

In a forthcoming paper [ChM06] we classify all triple-point defective linear systems L on ruled surfaces satisfying the assumptions of Theorem 4, and it follows from this classification that the linear system $|L - 3p|$ will contain the fibre of the ruling through p precisely with multiplicity two as a fixed component. In particular, the map β will automatically be generically finite.

Our method is based on the observation that, when (S, L) is triple-point defective, then at a general point $p \in S$ there exists a non-reduced scheme Z_p supported at the point, such that

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

By Serre's construction, this yields the existence of a rank 2 bundle $\mathcal{E}_p$ with first Chern class $L - K$, with a global section whose zero-locus is a subscheme of length at most 4, supported at p. Moreover the assumption $(L - K)^2 > 16$ implies that $\mathcal{E}_p$ is *Bogomolov unstable*, thus it has a destabilizing divisor A. By exploiting the properties of A and $B = L - K - A$, we obtain the result.

In a sort of sense, one of the main points missing for the proof of Segre's conjecture is a natural geometric construction for the non–reduced divisor which must be part of any defective linear system.
For double-point defective surfaces, the non–reduced component comes from contact loci of hyperplanes (see [ChC02]).
In our setting, the non–reduced component is essentially given by the effective divisor B above, which comes from a destabilizing divisor of the rank 2 bundle.
The result, applied to the blowing up of $\mathbb{P}^2$, leads to the following partial proof of Segre's conjecture on defective linear systems in the plane.

Corollary 6
Fix multiplicities $m_1 \leq m_2 \leq \cdots \leq m_n$. Let H denote the class of a line in $\mathbb{P}^2$ and assume that, for $p_1, \ldots, p_n$ general in $\mathbb{P}^2$, the linear system $M = rH - m_1p_1 - \cdots - m_np_n$ is defective, i.e.

$$\dim |M| > \max \left\{ -1, \binom{r+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} \right\}.$$

Let $\pi : S \longrightarrow \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points $p_2, \ldots, p_n$ and set $L := r\pi^*H - m_2E_2 - \cdots - m_nE_n$, where $E_i = \pi^*(p_i)$ is the i-th exceptional divisor. Assume that L is very ample on S, of the expected dimension $\binom{r+2}{2} - \sum_{i=2}^{n} \binom{m_i+1}{2}$, and that also $L - K$ is very ample on S, with $(L-K)^2 > 16$. Assume, finally, $m_1 \leq 3$.
Then $m_1 = 3$ and the general element of M is non-reduced. Moreover L embeds S as a ruled surface.

Proof: Just apply the Main Theorem 4 to the pair (S, L). $\square$

The reader can easily check that the previous result is exactly the translation of Segre's and Harbourne–Hirschowitz's conjectures on defective linear systems in the plane, for the case of a *minimally* defective system with lower multiplicity 3. The (-1)–curve predicted by Harbourne–Hirschowitz conjecture, in this situation, is just the pull-back of a line of the ruling.
Although the conditions "L and $L - K$ very ample" is not mild, we believe that the previous result could strengthen our believe in the general conjecture. Combining results in [Xu95] and [Laz97] Corollary.

2.6 one can give numerical conditions on r and the m_i such that L respectively $L-K$ are very ample.

The paper is organized as follows.
The case where β is not generically finite is pointed out in Theorem 8 in Section 2. In Section 3 we reformulate the problem as an h^1-vanishing problem. The Sections 4 to 7 are devoted to the proof of the main result: in Section 4 we use Serre's construction and Bogomolov instability in order to show that triple-point defectiveness leads to the existence of very special divisors A and B on our surface; in Section 5 we show that $|B|$ has no fixed component; in Section 6 we then list properties of B and we use these in Section 7 to classify the regular triple-point defective surfaces.

2. Triple components

In this section, we consider what happens when, in Diagram (1), the general fiber of β is positive-dimensional, in other words, when the general member of V_3 contains a triple component through p.
This case has been investigated (and essentially solved) in [Cas22], and then rephrased in modern language in [FrI01] and [BoC05].
Although not strictly necessary for the sequel, as our arguments do not make any use of the generic finiteness of β, (and so we will not assume this), for the sake of completeness we recall in this section some example and the classification of pairs (S, L) which are triple-point defective, and such that a general curve $L_p \in |L-3p|$ has a triple component through p.

The family $\mathcal{L}_3$ of pairs $(L, p) \in |L| \times S$ where $L \in |L-3p|$ has dimension bounded below by $\dim |L| - 4$, and in Remark 3 it has been pointed out that (S, L) is triple-point defective exactly when α is dominant and the bound is not attained.
Notice however that $\dim |L| - 4$ is *not* necessarily a bound for the dimension of the subvariety $V_3 \subset |L|$, the image of $\mathcal{L}_3$ under β. The following example (exploited in [LaM02]) shows that one may have $\dim(V_3) < \dim |L| - 4$ even when (S, L) is *not* triple-point defective.

Example 7 ((see [LaM02]))
Let S be the blowing up of $\mathbb{P}^2$ at 8 general points $q_1, \ldots, q_8$ and L

corresponds to the system of curves of degree nine in $\mathbb{P}^2$, with a triple point at each q_i.
$\dim |L| = 6$, but for $p \in S$ general, the unique divisor in $|L - 3p|$ coincides with the cubic plane curve through $q_1, \ldots, q_8, p$, counted three times. As there exists only a (non-linear) 1-dimensional family of such divisors in $|L|$, then $\dim(V_3) = 1 < \dim |L| - 4$. On the other hand, these divisors have a triple component, so that the general fibre of β has dimension 1, hence $\dim(\mathcal{L}_3) = 2 = \dim |L| - 4$.

The classification of triple-point defective pairs (S, L) for which the map β is not generically finite is the following.

Theorem 8
Suppose that (S, L) is triple-point defective. Then for $p \in S$ general, the general member of $|L - 3p|$ contains a triple component through p if and only if S lies in a threefold W which is a scroll in planes and moreover W is developable, i.e. the tangent space to W is constant along the planes.

Proof: (HINT) First, since we assume that S is triple-point defective and embedded in $\mathbb{P}^r$ via L, then the hyperplanes π that meet S in a divisor $H = S \cap \pi$ with a triple point at a general $p \in S$, intersect in a $\mathbb{P}^4$. Thus we may project down S to $\mathbb{P}^5$ and work with the corresponding surface.
In this setting, through a general $p \in S$ one has only one hyperplane π with a triple contact, and π has a triple contact with S along the fibre C of β. Thus V_3 is a curve.
If H', H'' are two consecutive infinitesimally near points to H on V_3, then C also belongs to $H \cap H' \cap H''$. Thus C is a plane curve and S is fibred by a 1-dimensional family of plane curves. This determines the threefold scroll W.
The tangent line to V_3 determines in $(\mathbb{P}^5)^*$ a pencil of hyperplanes which are tangent to S at any point of C, since this is the infinitesimal deformation of a family of hyperplanes with a triple contact along any point of C. Thus there is a $\mathbb{P}^4 = H_C$ which is tangent to S along C.
Assume that C is not a line. Then C spans a $\mathbb{P}^2 = \pi_C$ fibre of W, moreover the tangent space to W at a general point of C is spanned

by π_C and $T_{S,P}$, hence it is constantly equal to H_C. Since C spans π_C, then it turns out that the tangent space to W is constant at any point of π_C, i.e. W is developable.
When C is a line, then arguing as above one finds that all the tangent planes to S along C belong to the same $\mathbb{P}^3$. This is enough to conclude that S sits in some developable 3-dimensional scroll.
Conversely, if S is contained in the developable scroll W, then at a general point p, with local coordinates x, y, the tangent space t to W at p contains the derivatives $p, p_x, p_y, p_{xx}, p_{xy}$ (here x is the direction of the tangent line to C). Thus the $\mathbb{P}^4$ spanned by t, p_{yy} intersects S in a triple curve along C. $\square$

3. The Equimultiplicity Ideal

If L_p is a curve in $|L - 3p|$ we denote by $f_p \in \mathbb{C}\{x_p, y_p\}$ an equation of L_p in local coordinates x_p and y_p at p. If $\mathrm{mult}_p(L_p) = 3$, the ideal sheaf $\mathcal{J}_{Z_p}$ whose stalk at p is the equimultiplicity ideal

$$\mathcal{J}_{Z_p,p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p\rangle^3$$

of f_p defines a zero-dimensional scheme $Z_p = Z_p(L_p)$ concentrated at p, and the tangent space $T_{(L_p,p)}(\mathcal{L}_3)$ of $\mathcal{L}_3$ at (L_p, p) satisfies (see [Mar07] Example 10)

$$T_{(L_p,p)}(\mathcal{L}_3) \cong \big(H^0\big(S, \mathcal{J}_{Z_p}(L_p)\big)/H^0(S, \mathcal{O}_S)\big) \oplus \mathcal{K},$$

where $\mathcal{K}$ is zero unless L_p is unitangential at p, in which case $\mathcal{K}$ is a one-dimensional vector space.
In particular, $\mathcal{L}_3$ is smooth at (L_p, p) of the expected dimension (see [Mar07] Proposition 11)

$$\mathrm{expdim}(\mathcal{L}_3) = \dim|L| - 4$$

as soon as

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) = 0.$$

We thus have the following proposition.

Proposition 9
Let S be regular and suppose that α is surjective, then (S, L) is not

triple-point defective if

$$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) = 0$$

for general $p \in S$ *and* $L_p \in |L|$ *with* $\mathrm{mult}_p(L_p) = 3$.
Moreover, if L *is non-special the above* h^1*-vanishing is also necessary for the non-triple-point-defectiveness of* (S, L).

4. The Basic Construction

> *From now on we assume that for* $p \in S$ *general* $\exists\, L_p \in |L|$ *s.t.*
> $$h^1\big(S, \mathcal{J}_{Z_p}(L)\big) \neq 0.$$

Then by Serre's construction for a subscheme $Z_p' \subseteq Z_p$ with ideal sheaf $\mathcal{J}_p = \mathcal{J}_{Z_p'}$ of minimal length such that $h^1\big(S, \mathcal{J}_p(L)\big) \neq 0$ there is a rank two bundle $\mathcal{E}_p$ on S and a section $s \in H^0(S, \mathcal{E}_p)$ whose 0-locus is Z_p', giving the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E}_p \to \mathcal{J}_p(L-K) \to 0. \tag{2}$$

The Chern classes of $\mathcal{E}_p$ are

$$c_1(\mathcal{E}_p) = L - K \quad \text{and} \quad c_2(\mathcal{E}_p) = \mathrm{length}(Z_p').$$

Moreover, Z_p' is automatically a complete intersection.
We would now like to understand what $\mathcal{J}_p$ is depending on $\mathrm{jet}_3(f_p)$, which in suitable local coordinates will be one of those in Table (3). For this we first of all note that the very ample divisor L separates all subschemes of Z_p of length at most two. Thus Z_p' has length at least 3, and due to Lemma 10 below we are in one of the following situations:

$\mathrm{jet}_3(f_p)$	$\mathcal{J}_{Z_p,p}$	$\mathrm{length}(Z_p)$	$\mathcal{J}_p$	$c_2(\mathcal{E}_p)$
$x_p^3 - y_p^3$	$\langle x_p^2, y_p^2\rangle$	4	$\langle x_p^2, y_p^2\rangle$	4
$x_p^2 y_p$	$\langle x_p^2, x_p y_p, y_p^3\rangle$	4	$\langle x_p, y_p^3\rangle$	3
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3\rangle$	5	$\langle x_p^2, y_p^2\rangle$	4
x_p^3	$\langle x_p^2, x_p y_p^2, y_p^3\rangle$	5	$\langle x_p, y_p^3\rangle$	3

(3)

Lemma 10
If $f \in R = \mathbb{C}\{x,y\}$ *with* $\mathrm{jet}_3(f) \in \{x^3-y^3, x^2y, x^3\}$, *and if* $I = \langle g,h\rangle \lhd R$ *such that* $\dim_{\mathbb{C}}(R/I) \geq 3$ *and* $\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle + \langle x,y\rangle^3 \subseteq I$, *then we may assume that we are in one of the following cases:*

(a) $I = \langle x^2, y^2\rangle$ *and* $\mathrm{jet}_3(f) \in \{x^3-y^3, x^3\}$, *or*
(b) $I = \langle x, y^3\rangle$ *and* $\mathrm{jet}_3(f) \in \{x^2y, x^3\}$.

Proof: If $>$ is any *local degree* ordering on R, then the Hilbert-Samuel functions of R/I and of $R/L_>(I)$ coincide, where $L_>(I)$ denotes the leading ideal of I (see e.g. [GrP02] Proposition 5.5.7). In particular, $\dim_{\mathbb{C}}(R/I) = \dim_{\mathbb{C}}(R/L_>(I))$ and thus

$$L_>(I) \in \left\{\langle x^2, xy^2, y^3\rangle, \langle x^2, xy, y^2\rangle, \langle x^2, xy, y^3\rangle, \langle x^2, y^2\rangle, \langle x, y^3\rangle\right\},$$

since $\langle x^2, xy^2, y^3\rangle \subset I$.
Taking $>$, for a moment, to be the local degree ordering on R with $y > x$ we deduce at once that I does not contain any power series with a linear term in y. For the remaining part of the proof $>$ will be the local degree ordering on R with $x > y$.
1st Case: $L_>(I) = \langle x^2, xy^2, y^3\rangle$ or $L_>(I) = \langle x^2, xy, y^2\rangle$. Thus the graph of the slope $H^0_{R/I}$ of the Hilbert-Samuel function of R/I would be as shown in Figure 1, which contradicts the fact that I is a complete intersection due to [Iar77] Theorem 4.3.

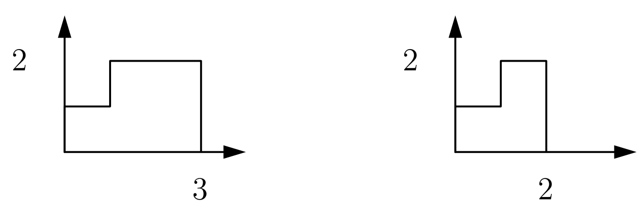

FIGURE 1. The graphs of $H^0_{R/\langle x^2,xy^2,y^3\rangle}$ respectively of $H^0_{R/\langle x^2,xy,y^2\rangle}$.

2nd Case: $L_>(I) = \langle x^2, xy, y^3\rangle$. Then we may assume

$$g = x^2 + \alpha\cdot y^2 + h.o.t. \quad \text{and} \quad h = xy + \beta\cdot y^2 + h.o.t..$$

Since $x^2 \in I$ there are power series $a, b \in R$ such that

$$x^2 = a\cdot g + b\cdot h.$$

Thus the leading monomial of a is one, a is a unit and $g \in \langle x^2, h\rangle$. We may therefore assume that $g = x^2$. Moreover, since the intersection multiplicity of g and h is $\dim_{\mathbb{C}}(R/I) = 4$, g and h cannot have a common tangent line in the origin, i. e. $\beta \neq 0$. Thus, since $g = x^2$, we may assume that $h = xy + y^2 \cdot u$ with $u = \beta + h.o.t$ a unit.
In new coordinates $\widetilde{x} = x \cdot \sqrt{u}$ and $\widetilde{y} = y \cdot \frac{1}{\sqrt{u}}$ we have

$$I = \langle \widetilde{x}^2, \widetilde{x}\widetilde{y} + \widetilde{y}^2\rangle.$$

Note that by the coordinate change $\mathrm{jet}_3(f)$ only changes by a constant, that $\frac{\partial f}{\partial \widetilde{x}}, \frac{\partial f}{\partial \widetilde{y}} \in I$ and that $\langle \widetilde{x}, \widetilde{y}\rangle^3 \subset I$, but $\widetilde{x}\widetilde{y}, \widetilde{y}^2 \notin I$. Thus $\mathrm{jet}_3(f) = x^3$. Setting now $\bar{x} = \widetilde{x}$ and $\bar{y} = \widetilde{x} + 2\widetilde{y}$, then $\bar{y}^2 = \widetilde{x}^2 + 4 \cdot (\widetilde{x}\widetilde{y} + \widetilde{y}^2) \in I$ and thus, considering colengths,

$$I = \langle \bar{x}^2, \bar{y}^2\rangle.$$

Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning.
<u>3rd Case:</u> $L_>(I) = \langle x^2, y^2\rangle$. Then we may assume

$$g = x^2 + \alpha \cdot xy + h.o.t. \quad \text{and} \quad h = y^2 + h.o.t.$$

As in the second case we deduce that w.l.o.g. $g = x^2$ and thus $h = y^2 \cdot u$, where u is a unit. But then $I = \langle x^2, y^2\rangle$.
<u>4th Case:</u> $L_>(I) = \langle x, y^3\rangle$. Then we may assume

$$g = x + h.o.t. \quad \text{and} \quad h = y^3 + h.o.t.$$

since there is no power series in I involving a linear term in y. In new coordinates $\widetilde{x} = g$ and $\widetilde{y} = y$ we have

$$I = \big\langle \widetilde{x}, \widetilde{h}\big\rangle,$$

and we may assume that $\widetilde{h} = \widetilde{y}^3 \cdot u$, where u is a unit only depending on $\widetilde{y}$. Hence, $I = \langle \widetilde{x}, \widetilde{y}^3\rangle$. Moreover, the 3-jet of f does not change with respect to the new coordinates, so that we may assume we worked with these from the beginning. $\square$

$$\boxed{\text{From now on we assume that } (L-K)^2 > 16.}$$

Thus

$$c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0,$$

and hence $\mathcal{E}_p$ is Bogomolov unstable. The Bogomolov instability implies the existence of a unique divisor A_p which destabilizes $\mathcal{E}_p$. (See e. g. [Fri98] Section 9, Corollary 2.) In other words, setting $B_p = L - K - A_p$, i. e.

$$A_p + B_p = L - K, \tag{4}$$

there is an immersion

$$0 \to \mathcal{O}_S(A_p) \to \mathcal{E}_p \tag{5}$$

where $(A_p - B_p)^2 \geq c_1(\mathcal{E}_p)^2 - 4 \cdot c_2(\mathcal{E}_p) > 0$ and $(A_p - B_p).H > 0$ for every ample H. From this we deduce the following properties:

(a) $\mathcal{E}_p(-A_p)$ has a section that vanishes along a subscheme $\widetilde{Z}_p$ of codimension 2, and

$$A_p.B_p \leq \text{length}(Z'_p). \tag{6}$$

The previous immersion gives rise to a short exact sequence:

$$0 \to \mathcal{O}_S(A_p) \to \mathcal{E}_p \to \mathcal{J}_{\widetilde{Z}_p}(B_p) \to 0. \tag{7}$$

(b) The divisor B_p is effective and we may assume that $Z'_p \subset B_p$.

(c) $A_p - B_p$ is big, and hence $\dim\big(|k\cdot(A_p - B_p)|\big) = const\cdot k^2 + o(k) >$ 0 for $k >> 0$. In particular

$$(A_p - B_p).M > 0 \tag{8}$$

if M is big and nef or if M is an irreducible curve with $M^2 \geq 0$ or if M is effective without fixed part.

(d) A_p is big.

Proof: (a) Sequence (7) is a consequence of Serre's construction. The first assertion now follows from Sequence (7), and Equation (6) is a consequence of

$$(A_p - B_p)^2 \geq c_1(\mathcal{E}_p)^2 - c_2(\mathcal{E}_p) = (A_p + B_p)^2 - 4 \cdot \text{length}(Z'_p).$$

(b) Observe that $\big(2A_p - (L - K)\big).H > 0$ for any ample line bundle H, and thus

$$-A_p.H < -\frac{(L - K_S).H}{2} < 0.$$

Thus $H^0\big(\mathcal{O}_S(-A_p)\big) = 0$ and twisting the sequence (2) with $-A_p$ we are done.

(c) Since $(A_p - B_p)^2 > 0$ and $(A_p - B_p).H > 0$ for some ample H Riemann-Roch shows that $A_p - B_p$ is big, i. e. $\dim\big(|k\cdot(A_p - B_p)|\big)$ grows with k^2. The remaining part follows from Lemma 11.

(d) This follows since $A_p - B_p$ is big and B_p is effective.

□

Lemma 11

Let R be a big divisor. If M is big and nef or if M is an irreducible curve with $M^2 \geq 0$ or if M is an effective divisor without fixed component, then $R.M > 0$.

Proof: If R is big, then $\dim |k \cdot R|$ grows with k^2. Thus for $k >> 0$ we can write $k \cdot R = N' + N''$ where N' is ample and N'' effective (possibly zero). To see this, note that for $k >> 0$ we can write $|k \cdot R| = |N'| + N''$, where N'' is the fixed part of $|k \cdot R|$ and $N' \cap C \neq \emptyset$ for every irreducible curve C. Then apply the Nakai-Moishezon Criterion to N' (see also [Tan04]).

Analogously, if M is big and nef, for $j >> 0$ we can write $j \cdot M = M' + M''$ where M' is ample and M'' is effective. Therefore,

$$R.M = \frac{1}{kj} \cdot \big(N'.M' + N'.M'' + N''.M\big) > 0,$$

since $N'.M' > 0$, $N'.M'' \geq 0$ and $N''.M \geq 0$.

Similarly, if M is irreducible and has non-negative self-intersection, then

$$R.M = \frac{1}{k} \cdot (N'.M + N''.M) > 0.$$

And if M is effective without fixed component, we can apply the previous argument to every component of M. □

Now let p move freely in S. Accordingly the scheme Z_p' moves, hence the effective divisor B_p containing Z_p' moves in an algebraic family $\mathcal{B} \subseteq |B|_a$ which is the closure of $\{B_p \mid p \in S, L_p \in |L - 3p|, \text{ both general}\}$ and which covers S. A priori this family $\mathcal{B}$ might have a *fixed part* C, so that for general $p \in S$ there is an effective divisor D_p moving in a fixed-part free algebraic family $\mathcal{D} \subseteq |D|_a$ such that

$$B_p = C + D_p.$$

Whenever we only refer to the algebraic class of A_p respectively B_p respectively D_p we will write A respectively B respectively D for short.

> For these considerations we assume, of course, that $\operatorname{length}(Z'_p)$ is constant for $p \in S$ general, so either $\operatorname{length}(Z'_p) = 3$ or $\operatorname{length}(Z'_p) = 4$.

5. $C = 0$.

Our first aim is to show that actually $C = 0$ (see Lemma 16). But in order to do so we first have to consider the boundary case that $A_p.B_p = \operatorname{length}(Z'_p)$.

Proposition 12
If $A_p.B_p = \operatorname{length}(Z'_p)$, then there exists a non-trivial global section $0 \neq s \in H^0\big(B_p, \mathcal{J}_{Z'_p/B_p}(A_p)\big)$ whose zero-locus is Z'_p.
In particular, $A_p.D_p = A_p.B_p = \operatorname{length}(Z'_p)$ and $A_p.C = 0$.

Proof: By Equation (7) we have

$$A_p.B_p = \operatorname{length}(Z'_p) = c_2(\mathcal{E}_p) = A_p.B_p + \operatorname{length}\big(\widetilde{Z}_p\big).$$

Thus $\widetilde{Z}_p = \emptyset$.
If we merge the sequences (2), (7), and the structure sequence of B twisted by B we obtain the following exact commutative diagram in Figure 2, where $\mathcal{O}_{B_p}(B_p) = \mathcal{J}_{Z'_p/B_p}(A_p + B_p)$, or equivalently $\mathcal{O}_{B_p} = \mathcal{J}_{Z'_p/B_p}(A_p)$. Thus from the rightmost column we get a non-trivial global section, say s, of this bundle which vanishes precisely at Z'_p, since Z'_p is the zero-locus of the monomorphism of vector bundles $\mathcal{O}_S \hookrightarrow \mathcal{E}_p$. However, since p is general we have that $p \notin C$ and thus the restriction $0 \neq s_{|D_p} \in H^0\big(D_p, \mathcal{J}_{Z'_p/D_p}(A_p)\big)$ and it still vanishes precisely at Z'_p. Thus $A_p.D_p = \operatorname{length}(Z'_p) = A_p.B_p$, and $A_p.C = A_p.B_p - A_p.D_p = 0$. □

Lemma 13
$A_p.B_p \geq 1 + B_p^2$.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_S(A_p) \arrow[r] \arrow[d] & \mathcal{E}_p \arrow[r] \arrow[d] & \mathcal{O}_S(B_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_S(A_p) \arrow[r] \arrow[d] & \mathcal{J}_{Z'_p/S}(A_p+B_p) \arrow[r] \arrow[d] & \mathcal{O}_{B_p}(B_p) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

FIGURE 2. The diagram showing $\mathcal{O}_{B_p} = \mathcal{J}_{Z'_p/B_p}(A_p)$.

Proof: Let $B = P + N$ be a Zariski decomposition of B, i. e. P and N are effective $\mathbb{Q}$-divisors such that in particular P is nef, $P.N = 0$ and $N^2 < 0$ unless $N = 0$.

If $N \neq 0$, then

$$0 < (A + B).N = A.N + N^2,$$

since $A + B$ is very ample and N is effective. Moreover, since P is nef and $A - B$ big we have $(A - B).P \geq 0$ and hence

$$A.P \geq B.P = P^2.$$

Combining these two inequalities we get

$$A.B = A.P + A.N > P^2 - N^2 > P^2 + N^2 = B^2.$$

If $N = 0$, then B is nef and, therefore, $B^2 \geq 0$. If, actually $B^2 > 0$, then B is big and nef, so that by (8) $(A - B).B > 0$. While if $B^2 = 0$ then

$$B^2 = 0 < B.(A + B) = A.B$$

since $A + B$ is very ample and B is effective. $\square$

Lemma 14

Let $p \in S$ be general and suppose $\text{length}(Z'_p) = 4$.

(a) *If D_p is irreducible, then* $\dim(\mathcal{D}) \geq 2$ *and* $D_p^2 \geq 3$.

(b) *If D_p is reducible but the part containing p is reduced, then either D_p has a component singular in p and $D_p^2 \geq 5$ or at least two components of D_p pass through p and $D_p^2 \geq 2$.*

(c) *If $D_p^2 \leq 1$, then $D_p = k \cdot E_p$ where $k \geq 2$, E_p is irreducible and $E_p^2 = 0$. In particular, $D_p^2 = 0$.*

Proof: (a) If D_p is irreducible, then $\dim(\mathcal{D}) \geq 2$, since D_p, containing Z'_p, is singular in p by Table (3) and since $p \in S$ is general. If through $p \in S$ general and a general $q \in D_p$ there is another $D' \in \mathcal{D}$, then due to the irreducibility of D_p

$$D_p^2 = D_p.D' \geq \mathrm{mult}_p(D_p) + \mathrm{mult}_q(D_p) \geq 3.$$

Otherwise, $\mathcal{D}$ is a two-dimensional involution whose general element is irreducible, so that by [ChC02] Theorem 5.10 $\mathcal{D}$ must be a linear system. This, however, contradicts the Theorem of Bertini, since the general element of $\mathcal{D}$ would be singular.

(b) Suppose $D_p = \sum_{i=1}^k E_{i,p}$ is reducible but the part containing p is reduced. Since D_p has no fixed component and p is general, each $E_{i,p}$ moves in an at least one-dimensional family. In particular $E_{i,p}^2 \geq 0$.

If some $E_{i,p}$, say $i = 1$, would be singular in p for $p \in S$ general we could argue as above that $E_{1,p}^2 \geq 3$. Moreover, either $E_{2,p}$ is algebraically equivalent to $E_{1,p}$ and $E_{2,p}^2 \geq 3$, or $E_{1,p}$ and $E_{2,p}$ intersect properly, since both vary in different, at least one-dimensional families. In any case we have

$$D_p^2 \geq (E_{1,p} + E_{2,p})^2 \geq 5.$$

Otherwise, at least two components, say $E_{1,p}$ and $E_{2,p}$ pass through p, since D_p is singular in p and no component passes through p with higher multiplicity. Hence, $E_{1,p}.E_{2,p} \geq 1$ and therefore

$$D_p^2 \geq 2 \cdot E_{1,p}.E_{2,p} \geq 2.$$

(c) From the above we see that D_p is not reduced in p. Let therefore $D_p \equiv_a kE_p + E'$ where $k \geq 2$, E_p passes through p and E' does not contain any component algebraically equivalent to E_p.

Suppose $E' \neq 0$. Since D_p has no fixed component both, E_p and E' vary in an at least one dimensional family covering S and must therefore intersect properly. In particular, $E_p.E' \geq 1$ and $1 \geq D_p^2 \geq 2k \cdot E_p.E' \geq 4$. Thus, $E' = 0$.

We therefore may assume that $D_p = kE_p$ with $k \geq 2$. Then $0 \leq E_p^2 = \frac{1}{k^2} \cdot D_p^2 \leq \frac{1}{4}$, which leaves only the possibility $E_p^2 = 0$, implying also $D_p^2 = 0$.

□

Lemma 15

Suppose that $R \subset S$ is an irreducible curve.

(a) *If $(L-K).R \in \{1,2\}$, then R is smooth, rational and $R^2 \leq (L-K).R - 3 \leq -1$.*

(b) *If $(L-K).R = 3$, then $R^2 \leq 0$, and either R is a plane cubic or it is a smooth rational space curve.*

Proof: Note that S is embedded in some $\mathbb{P}^n$ via $|L-K|$ and that $\deg(R) = (L-K).R$ is just the degree of R as a curve in $\mathbb{P}^n$. Moreover, by the adjunction formula we know that

$$p_a(R) = \frac{R^2 + R.K}{2} + 1,$$

and since L is very ample we thus get

$$1 \leq L.R = (L-K).R + R.K = (L-K).R + 2 \cdot \big(p_a(R) - 1\big) - R^2. \quad (9)$$

(a) If $\deg(R) \in \{1,2\}$, then R must be a smooth, rational curve. Thus we deduce from (9)

$$R^2 \leq (L-K).R - 3.$$

(b) If $\deg(R) = 3$, then R is either a plane cubic or a smooth space curve of genus 0. If $p_a(R) = 1$ then actually $L.R \geq 3$ since otherwise $|L|$ would embed R as a rational curve of degree 1 resp. 2 in some projective space. In any case we are therefore done with (9).

□

Lemma 16

$C = 0$.

Proof: Suppose $C \neq 0$ and r is the number of irreducible components of C. Since $\mathcal{D}$ has no fixed component we know by (6) that $(A-B).D > 0$, so that

$$A.D \geq B.D + 1 = D.C + D^2 + 1 \tag{10}$$

or equivalently

$$D.C \leq A.D - D^2 - 1. \tag{11}$$

Moreover, since $A+B$ is very ample we have $r \leq (A+B).C = A.C + D.C + C^2$ and thus

$$A.C + D.C = (A+B).C - C^2 \geq r - C^2. \tag{12}$$

1st Case: $C^2 \leq 0$. Then (12) together with (10) gives

$$A.B = A.C + A.D \geq A.C + D.C + D^2 + 1 \geq r + (-C^2) + D^2 + 1 \geq 2, \tag{13}$$

or the slightly stronger inequality

$$\operatorname{length}(Z'_p) \geq A.B \geq (A+B).C + (-C^2) + D^2 + 1. \tag{14}$$

2nd Case: $C^2 > 0$. Then by Lemma 13 simply

$$\operatorname{length}(Z'_p) \geq A.B \geq B^2 + 1 = D^2 + 2 \cdot C.D + C^2 + 1 \geq 2. \tag{15}$$

Since all the summands involved in the right hand side of (13) and (15) are non-negative and since by Lemma 14 the case $D^2 = 1$ cannot occur when $\operatorname{length}(Z'_p) = 4$, we are left considering the cases shown in Figure 3, where for the additional information (the last four columns) we take Proposition 12 and Lemma 14 into account.
Let us first and for a while consider the situation $\operatorname{length}(Z'_p) = 4$ and $D^2 = 0$, so that by Lemma 14 $D = kE$ for some irreducible curve E with $k \geq 2$ and $E^2 = 0$. Applying Lemma 15 to E we see that $(A+B).E \geq 3$, and thus

$$6 \leq 3k \leq (A+B).D = A.D + C.D. \tag{16}$$

If in addition $A.D \leq 4$, then (11) leads to

$$6 \leq A.D + C.D \leq 4 + C.D \leq 7, \tag{17}$$

which is only possible for $k = 2$, $C.E = 1$ and

$$C.D = k \cdot C.E = 2. \tag{18}$$

This outrules Case 12.

	length(Z'_p)	D^2	C^2	$C.D$	r	$A.B$	$A.D$	$A.C$	D
1)	4	0	-2		1	4	4	0	$kE, k \geq 2$
2)	4	0	-1		2	4	4	0	$kE, k \geq 2$
3)	4	0	0		3	4	4	0	$kE, k \geq 2$
4)	4	0	-1		1	$3, 4$			$kE, k \geq 2$
5)	4	2	0		1	4	4	0	
6)	4	0	0		2	$3, 4$			$kE, k \geq 2$
7)	4	0	0		1	$2, 3, 4$			$kE, k \geq 2$
8)	3	0	-1		1	3	3	0	
9)	3	0	0		2	3	3	0	
10)	3	1	0		1	3	3	0	
11)	3	0	0		1	$2, 3$			
12)	4	0	1	1		4	4	0	$kE, k \geq 2$
13)	4	2	1	0		4	4	0	
14)	4	0	1	0		$2, 3, 4$			$kE, k \geq 2$
15)	4	0	2	0		$3, 4$			$kE, k \geq 2$
16)	3	1	1	0		3	3	0	
17)	3	0	1	0		$2, 3$			

FIGURE 3. The cases to be considered.

In Cases 1, 2 and 3 we have $A.D = 4$, and we can apply (18), which by (12) then gives the contradiction

$$2 = A.C + C.D \geq r - C^2 = 3.$$

If, still under the assumption $\text{length}(Z'_p) = 4$ and $D^2 = 0$, we moreover assume $2 \geq C^2 \geq 0$ then by Lemma 13

$$3 \geq B^2 = 2 \cdot C.D + C^2 \geq 2 \cdot C.D \geq 0,$$

and thus $C.D \leq 1$ and $C.D + C^2 \leq 2$, which due to (16) implies $A.D \geq 5$. But then by Proposition 12 we have $A.B \leq 3$ and hence $A.C = A.B - A.D \leq -2$, which leads to the contradiction

$$(A + B).C = A.C + D.C + C^2 \leq 0, \tag{19}$$

since $A + B$ is very ample. This outrules the Cases 6, 7, 14 and 15.
In Case 4 Lemma 15 applied to C shows

$$2 \leq (A + B).C = A.C + D.C + C^2. \tag{20}$$

If in this situation $A.B = 4$, then Proposition 12 shows $A.C = 0$ and $A.D = A.B = 4$, and therefore (18) leads a contradiction, since the right hand side of Equation (20) is $A.C + D.C + C^2 = 0 + 2 - 1 = 1$. We, therefore, conclude that $A.B = 3$, and as above we get from Lemma 13

$$2 \geq B^2 = 2 \cdot C.D + C^2 = 2k \cdot C.E - 1 \geq 4 \cdot C.E - 1 \geq -1,$$

which is only possible for $C.E = C.D = 0$. But then (20) implies $A.C \geq 3$, and since A is big and D has no fixed component Lemma 11 gives the contradiction

$$1 \leq A.D = A.B - A.C \leq 0.$$

This finishes the cases where $\text{length}(Z'_p) = 4$ and $D^2 = 0$.
In Cases 5, 10 and 11 we apply Lemma 15 to the irreducible curve C with $C^2 = 0$ and find

$$(A + B).C \geq 3.$$

In Cases 5 and 10 Equation (14) then gives the contradiction

$$4 \geq A.B \geq 3 - C^2 + D^2 + 1 \geq 5,$$

and similarly in Case 11 we get

$$3 \geq A.B \geq 3 - C^2 + D^2 + 1 = 4.$$

In very much the same way we get in Case 8

$$(A + B).C \geq 2$$

and the contradiction

$$3 \geq A.B \geq 2 - C^2 + D^2 + 1 = 4.$$

Let us next have a look at the Cases 16 and 17. Consider the decomposition of the general $D = \sum_{i=1}^{s} E_i$ into irreducible components, none of which is fixed. In Case 16 we have $D^2 = 0$, and thus $E_i.E_j = 0$ for all i, j, while in Case 17 we have $D^2 = 1$ and we thus may assume $E_1^2 = 1$ and $E_i.E_j = 0$ for all $(i, j) \neq (1, 1)$. Moreover, in both cases $C.D = 0$ and thus $C.E_i = 0$ for all i. Applying Lemma 15 to E_i we find

$$A.E_i = (A + B).E_i - E_1.E_i \geq 3,$$

and by (12) we get

$$A.C = A.C + D.C \geq r - C^2 \geq 0. \tag{21}$$

But then

$$3 \geq A.B = A.C + \sum_{i=1}^{s} A.E_i \geq 3s,$$

which implies $s = 1$ and $A.C = 0$. From (21) we deduce that $r = C^2 = 1$, and thus C is irreducible with $C^2 = 1$. Similarly in Case 13 we have by (12)

$$0 = A.C + D.C \geq r - C^2 = r - 1 \geq 0,$$

and again C is irreducible with $C^2 = 1$. Applying now Lemma 15 to C we get in each of these three cases the contradiction

$$4 \leq (A + B).C = A.C + D.C + C^2 = 1.$$

This outrules the Cases 13, 16, and 17.

It remains to consider Case 9. Here we deduce from (14) that

$$2 \geq (A + B).C \geq r = 2,$$

and hence

$$2 = (A + B).C = A.C + D.C + C^2 = D.C.$$

But then Lemma 13 leads to the final contradiction

$$2 = A.B - 1 \geq B^2 = D^2 + 2 \cdot D.C + C^2 = 4.$$

□

It follows that $B_p = D_p$, $\mathcal{B} = \mathcal{D}$, and that B_p is nef.

6. The General Case

Let us review the situation and recall some notation. We are considering a divisor L such L and $L-K$ are very ample with $(L-K)^2 > 16$, and such that for a general point $p \in S$ the general element $L_p \in |L-3p|$ has no triple component through p and that the equimultiplicity ideal of L_p in p in suitable local coordinates is one of the ideals in Table (3) – and for all p the ideals have the same length. Moreover, we know that there is an algebraic family $\mathcal{B} = \overline{\{B_p \mid p \in S\}} \subset |B|_a$ without fixed component such that for a general point $p \in S$

$$B_p \in |\mathcal{J}_{Z'_p/S}(L-K-A_p)|,$$

where Z'_p is the equimultiplicity scheme of L_p and A_p is the unique divisor linearly equivalent to $L-K-B_p$ such that B_p and A_p destabilize the vector bundle $\mathcal{E}_p$ in (2). Keeping these notations in mind we can now consider the two cases that either $\text{length}(Z'_p) = 4$ or $\text{length}(Z'_p) = 3$.

Proposition 17
Let $p \in S$ be general and suppose that $\text{length}(Z'_p) = 4$. *Then $B_p = E_p + F_p$, E_p and F_p are irreducible, smooth, elliptic curves, $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$, $A.E_p = A.F_p = 2$, $L.E_p = L.F_p = 3$, $A.B = 4$, $K.E_p = K.F_p = 0$, and $\exists\, s \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that $Z'_p = \{s = 0\}$.*
Moreover, neither $|E|_a$ and $|F|_a$ is a linear system, but they both induce an elliptic fibration with section on S over an elliptic curve.

Proof: Since $A^2 > 0$ we can apply the Hodge Index Theorem (see e.g. [BHPV04]), and since $(A+B)^2 \geq 17$ by assumption and $A.B \leq 4$ by Equation (6) we deduce

$$\begin{aligned} 16 \geq (A.B)^2 \geq A^2 \cdot B^2 = \big((A+B)^2 - 2A.B - B^2\big).B^2 \\ \geq (9 - B^2) \cdot B^2. \quad (22) \end{aligned}$$

In Section 5 we have shown that $B = D$ is nef, and thus Lemma 13 together with Equation (22) shows

$$0 \leq B^2 \leq 2. \qquad (23)$$

Then, however, Lemma 14 implies that B_p must be reducible.

Let us first consider the case that the part of B_p through p is reduced. Then by Lemma 14, Lemma 13, and Equations (6) and (23) we know that $B_p = E_p + F_p + R$, where E_p and F_p are irreducible and smooth in p. In particular, $E_p.F_p \geq 1$, and thus

$$2 \geq B^2 = E_p^2 + 2 \cdot E_p.F_p + F_p^2 + 2 \cdot (E_p + F_p).R + R^2$$
$$\geq 2 + 2 \cdot (E_p + F_p).R.$$

Since $E_p.F_p = 1$ and since the components E_p and F_p vary in at least one-dimensional families and R has no fixed component, $(E_p+F_p).R \geq 1$, unless $R = 0$. This would however give a contradiction, so $R = 0$. Therefore necessarily, $B_p = E_p + F_p$, $E_p.F_p = 1$, and $E_p^2 = F_p^2 = 0$. Then by Lemma 15 $(A + B).E_p \geq 3$ and $(A + B).F_p \geq 3$, so that

$$4 \geq A.B \geq (A + B).E_p + (A + B).F_p - B^2 \geq 4$$

implies $E_p.A_p = 2 = F_p.A_p$ and $(A+B).E_p = 3 = (A+B).F_p$. Applying Lemma 15 once more, we see that

$$p_a(E_p) \leq 1 \quad \text{and} \quad p_a(F_p) \leq 1. \tag{24}$$

We claim that in p the curve L_p can share at most with one of E_p or F_p a common tangent, and it can do so at most with multiplicity one. For this consider local coordinates (x_p, y_p) as in the Table (3). Since $\text{length}(Z_p') = 4$ we know that $\mathcal{J}_{Z_p',p} = \langle x_p^2, y_p^2\rangle$ does not contain x_py_p, and since $B_p = E_p + F_p \in |\mathcal{J}_{Z_p'}(L - K - A)|$, where E_p and F_p are smooth in p, we deduce that in local coordinates their equations are

$$x_p + a \cdot y_p + h.o.t. \quad \text{respectively} \quad x_p - a \cdot y_p + h.o.t.,$$

where $a \neq 0$. By Table (3) the local equation f_p of L_p has either $\text{jet}_3(f_p) = x_p^3$ and has thus no common tangent with either E_p or F_p, or $\text{jet}_3(f_p) = x_p^3 - y_p^3$ and it is divisible at most once by one of $x_p - ay_p$ or $x_p + ay_p$.

In particular, E_p can at most once be a component of L_p, and we deduce

$$E_p.K_S = E_p.L_p - E_p.A_p - E_p.B_p = E_p.L_p - 3 \geq \begin{cases} 0, & \text{if } E_p \not\subset L_p, \\ -1, & \text{if } E_p \subset L_p. \end{cases}$$

But then, since the genus is an integer,

$$p_a(E_p) = \frac{E_p^2 + E_p.K_S}{2} + 1 = \frac{E_p.K_S}{2} + 1 \geq 1,$$

in which case (24) gives $p_a(E_p) = 1$. This shows, in particular, that

$$K.E_p = 0 \quad \text{and} \quad L_p.E_p = 3.$$

By symmetry the same holds for F_p.
Since $E_p^2 = 0$ the family $|E|_a$ is a pencil and induces an elliptic fibration on S (see [Kei01] App. B.1). In particular, the generic element E_p in $|E|_a$ must be smooth (see e.g. [BHPV04] p. 110). And with the same argument the generic element F_p in $|F|_a$ is smooth.
Suppose now that $|E|_a$ is a linear system. Since $E.F = 1$ and for $q \in F_p$ general $E_q \cap F_p = \{q\}$ the linear system $|\mathcal{O}_{F_p}(E)|$ is a $\mathfrak{g}_1^1$ on the smooth curve F_p implying that F_p is rational contradicting $p_a(F_p) = 1$. Thus $|E|_a$ is not linear, and analogously $|F|_a$ is not.
It remains to consider the case that B_p is not reduced in p. Using the notation of the proof of Lemma 14 we write $B_p \equiv k \cdot E_p + E'$ with $k \geq 2$, E_p irreducible passing through p and E' not containing any component algebraically equivalent to E_p. We have seen there (see p. 18) that $E' \neq 0$ implies $B_p^2 \geq 4$ in contradiction to Lemma 13. We may therefore assume $B_p = k \cdot E_p$ with $E_p^2 \geq 0$. If $E_p^2 \geq 1$, then again $B_p^2 \geq 4$. Thus $E_p^2 = 0$. Applying Lemma 15 to E_p we get

$$3 \leq (A + B).E_p = A.E_p,$$

and hence the contradiction

$$4 \geq A.B = k \cdot A.E_p \geq 6.$$

Therefore, B_p must be reduced in p. □

Proposition 18
Let $p \in S$ be general and suppose that $\text{length}(Z_p') = 3$. *Then B_p is an irreducible, smooth, rational curve in the pencil $|B|_a$ with $B^2 = 0$, $A.B = 3$ and $\exists\ s \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s.*
In particular, $S \to |B|_a$ is a ruled surface and $2B_p$ is a fixed component of $|L - 3p|$.

Proof: Since $A^2 > 0$ we can apply the Hodge Index Theorem (see e.g. [BHPV04]), and since $(A+B)^2 \geq 17$ by assumption and $A.B \leq 3$ by Equation (6) we deduce

$$9 \geq (A.B)^2 \geq A^2 \cdot B^2 = \big((A+B)^2 - 2A.B - B^2\big) \geq (11 - B^2) \cdot B^2.$$

Since in Section 5 we have shown that B is nef, this inequality together with Lemma 13 implies

$$B^2 = 0. \tag{25}$$

Once we have shown that B_p is irreducible and reduced, we then know that $|B|_a$ is a pencil and induces a fibration on S whose fibres are the elements of $|B|_a$ (see [Kei01] App. B.1). In particular, the general element of $|B|_a$, which is B_p, is smooth (see [BHPV04] p. 110).
Let us therefore first show that B_p is irreducible and reduced. Since $\mathcal{B}$ has no fixed component we know for each irreducible component B_i of $B_p = \sum_{i=1}^r B_i$ that $B_i^2 \geq 0$, and hence by Lemma 15 that $(A+B).B_i \geq 3$. Thus by (6) and (25)

$$3 \cdot r \leq (A+B).B = A.B + B^2 = A.B \leq 3,$$

which shows that B_p is irreducible and reduced and that $A.B = 3$. Moreover, $(A+B).B = 3$, and Lemma 15 implies that

$$p_a(B_p) \leq 1. \tag{26}$$

Since $A.B = 3 = \text{length}(Z_p')$ Proposition 12 implies that there is a section $s_p \in H^0\big(B_p, \mathcal{O}_{B_p}(A_p)\big)$ such that Z_p' is the zero-locus of s_p, which is just $3p$. Note that for $p \in S$ general and $q \in B_p$ general we have $B_p = B_q$ since $|B|_a$ is a pencil, and thus by the construction of B_p and B_q we also have

$$A_p \sim_l L - K - B_p = L - K - B_q \sim_l A_q.$$

But if A_p and A_q are linearly equivalent, then so are the divisors s_p and s_q induced on the curve $B_p = B_q$. The curve B_p therefore contains a linear series $|\mathcal{O}_{B_p}(A_p)|$ of degree three which contains $3q$ for a general point $q \in B_p$. If B_p was an elliptic curve, then $|\mathcal{O}_{B_p}(A_p)|$ would necessarily have to be a $\mathfrak{g}_3^2$ embedding B_p as a plane curve of degree

three and the general point q would be an inflexion point. But that is clearly not possible. Thus

$$p_a(B_p) = 0,$$

and by the adjunction formula we get

$$K.B = 2p_a(B) - 2 - B^2 = -2. \tag{27}$$

Note also, that $Z'_p \subset B_p$ in view of Table (3) implies that B_p and L_p have a common tangent in p. Suppose that B_p and L_p have no common component, i. e. $B_p \not\subset L_p$, then

$$3 \leq \mathrm{mult}_p(B_p) \cdot \mathrm{mult}_p(L_p) < L.B = A.B + B^2 + K.B = 3 + K.B = 1,$$

which contradicts (27). Thus, B_p is at least once contained in L_p. Moreover, if $2B_p \not\subset L_p$ then by Table (3) $L'_p := L_p - B_p$ has multiplicity two in p, and it still has a common tangent with B_p in p, so that

$$3 \leq L'_p.B_p = L.B - B^2 = A.B + K.B = 3 + K.B = 1 \tag{28}$$

again is impossible. We conclude finally, that B_p is at least twice contained in L_p

Note finally, since $\dim |B|_a = 1$ there is a unique curve B_p in $|B|_a$ which passes through p, i. e. it does not depend on the choice of L_p, so that in these cases B_p respectively $2B_p$ is actually a fixed component of $|L - 3p|$. □

7. Regular Surfaces

Theorem 19 ("If S is regular, then S is a rationally ruled surface.") *More precisely, let S be a regular surface and L a line bundle on S such that L and $L - K$ are very ample. Suppose that $(L - K)^2 > 16$ and that for a general $p \in S$ the linear system $|L - 3p|$ contains a curve L_p which has no triple component through p, but such that $h^1\big(\mathcal{J}_{Z_p}(L)\big) \neq 0$ where Z_p is the equimultiplicity scheme of L_p at p.*
Then there is a rational ruling $\pi : S \to \mathbb{P}^1_{\mathbb{C}}$ of S such that L_p contains the fibre over $\pi(p)$ with multiplicity two.

Proof: Let us suppose that S is regular, so that each algebraic family is indeed a linear system, and let $p \in S$ be general.

The case $\mathrm{length}(Z'_p) = 4$ is excluded since by Proposition 17 the algebraic families $|E|_a$ and $|F|_a$ would have to be linear systems. Thus necessarily $\mathrm{length}(Z'_p) = 3$, and we are done by Proposition 18. $\square$

References

[BHPV04] Wolf Barth, , Klaus Hulek, Christian Peters, and Antonius Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 4, Springer, 2004.

[BoC05] Cristiano Bocci and Luca Chiantini, *Triple points imposing triple divisors and the defective hierarchy*, Projective Varieties with Unexpected Properties, De Gruyter, 2005, pp. 35–49.

[Cas22] Maria Castellani, *Sulle superficie i cui spazi osculatori sono biosculatori*, Rend. Accad. Lincei **XXXI** (1922), pp. 347–350.

[ChC02] Luca Chiantini and Ciro Ciliberto, *Weakly defective varieties*, Trans. Amer. Math. Soc. **354** (2002), pp. 151–178.

[ChM06] Luca Chiantini and Thomas Markwig, *Triple-point defective ruled surfaces*, Preprint Kaiserslautern (2007).

[Cil01] Ciro Ciliberto, *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math. **201** (2001), pp. 289–316.

[FrI01] Davide Franco and Giovanna Ilardi, *On multiosculating spaces*, Comm. in Alg. **29** (2001), no. 7, pp. 2961–2976.

[Fri98] Robert Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer, 1998.

[FuP00] Luis Fuentes and Manuel Pedreira, *The projective theory of ruled surfaces*, math.AG/0006204, June 2000.

[GrP02] Gert-Martin Greuel and Gerhard Pfister, *A* SINGULAR *introduction to commutative algebra*, Springer, 2002.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer, 1977.

[Iar77] Anthony Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **10** (1977), no. 188, pp. 1–97.

[Kei01] Thomas Keilen, *Families of curves with prescribed singularities*, Ph.D. thesis, Universität Kaiserslautern, 2001, http://www.mathematik.uni-kl.de/˜keilen/download/Thesis/thesis.ps.gz.

[LaM02] Antonio Lanteri and Raquel Mallavibarrena, *Jets of antimulticanonical bundles on Del Pezzo surfaces of degree* ≤ 2, Algebraic Geometry, A Volume in Memory of Paolo Francia, De Gruyter, 2002, pp. 258–276.

[Laz97] Robert Lazarsfeld, *Lectures on linear series*, Complex Algebraic Geometry (János Kollár, ed.), IAS/Park City Mathematics Series, no. 3, Amer. Math. Soc., 1997, pp. 161–219.

[Mar07] Thomas Markwig, *A note on equimultiple deformations*, arXive0705.3911, May 2007.

[Tan04] Sheng-Li Tan, *Effective behaviour of multiple linear systems*, Asian J. Math. **8**, (2004) no. 2, 287–304.

[Ter22] Alessandro Terracini, *Su due problemi concernenti la determinazione di alcune classi di superficie, considerate da G. Scorza e F. Palatini*, Atti Soc. Natur. e Matem. Modena **6** (1921-22), pp. 3–16.

[Xu95] Geng Xu, *Ample line bundles on smooth surfaces*, J. reine angew. Math. **469** (1995), 199–209.

Universitá degli Studi di Siena, Dipartimento di Scienze Matematiche e Informatiche, Pian dei Mantellini, 44 I – 53100 Siena

E-mail address: chiantini@unisi.it

URL: http://www.dsmi.unisi.it/newsito/docente.php?id=4

Universität Kaiserslautern, Fachbereich Mathematik, Erwin-Schrödinger-Strasse, D – 67663 Kaiserslautern

E-mail address: keilen@mathematik.uni-kl.de

URL: http://www.mathematik.uni-kl.de/~keilen